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Deformation of Lie algebras in a non-Chevalley basis and 'embedding' of q-algebras

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Abstract. A deformation scheme for Lie algebras in a basis which manifestly exhibits for $q=1$ the content of a singular subalgebra is presented. This scheme allows to build up 'embedding' chains of q-algebras which can be physically interesting. In the present paper the case when the rank of the subalgebra is one unit less than the rank of the algebra is studied and explicit constructions of the algebras are given in terms of q-bosonic and/or q-fermionic oscillators.

1. Introduction

The quantum algebras G_q or $U_q(G)$, i.e. the q-deformed universal enveloping algebra of a semi-simple Lie algebra G (see for instance [1] for a more precise definition) are actually a topic of very active research both in physics and mathematics. It is far beyond the aim of this paper to present even a short review of the many physical applications which have been proposed starting from different points of view. The motivations of this study lie on the fact that the underlying idea in some applications of q-algebras is to use a q-deformed algebra instead of a Lie algebra to realize a *generalized dynamical symmetry*. It is well known that the generalized dynamical symmetry in many models of nuclear, hadronic, molecular and chemical physics is displayed through embedding chains of algebras of the type

$$G_0 \supset G_1 \supset \ldots \supset SO(3) \supset SO(2) \tag{1.1}$$

where $SO(3)$ describes the angular momentum and, usually, the Lie algebras are realized in terms of bosonic creation-annihilation operators. An essential step to carry forward the program of application of q-algebras as *generalized dynamical symmetry* is to dispose on a formalism which allows to build up chains analogous to equation (1.1) replacing the Lie algebras by the deformed ones.

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The simplest, not trivial, chain is the q-analogous of the embedding chain of the Elliott model [2], i.e.

$$SU_q(3) \supset SO_q(3).$$

In the Elliott model $SO(3)$ is the three-dimensional principal subalgebra of $SU(3)$. Quite recently Van der Jeugt [3] has investigated the existence of three-dimensional principal q-subalgebra for $Gl_q(n+1)$, showing that such a subalgebra exists only for $n=2$ when the algebraic relations are restricted to the symmetric representations, but the coproduct of $Gl_q(3)$ *does not induce* a coproduct in the three-dimensional principal subalgebra. It is useful to emphasize that the definition of the coproduct is essential to define the tensor product of spaces.

In [4] we tried to solve the problem the other way around: to define a *true* $SO_q(3)$, i.e. a deformed $SO(3)$ in which a coproduct on the generators is defined, and to build up a deformed structure of the type $Gl_q(3)$ or $SU_q(3)$. Indeed it has been shown that a 'deformed $Gl(3)$' can be obtained but, besides some ambiguity in the procedure, it was not clear how to impose on it the Hopf structure. The problem has been recently tackled by Quesne [5], which has constructed a deformed $SU(3)$ algebra in terms of q-boson operators transforming as vector under $SO_q(3)$, but many questions remain open in this approach, although quite interesting in view of its possible generalization to more general irreducible q-tensor operators. In particular also in this approach it is not clear how to endow the 'deformed $U(3)$' algebra with a Hopf structure.

In a very simple and general way, G_q is well mathematically defined in the Chevalley–Cartan basis, see [1], and we believe that the very root of the problem lies on the fact that this basis is not suitable to discuss embedding of subalgebras except the trivial ones. So we present here an alternative deformation scheme which can be useful to discuss 'embedding' chains of the type of (1.1). Let us immediately emphasize that really the word 'embedding' is used in some loose sense as we will show that the obtained 'deformed algebra G' is not the same as the usual G_q. In particular:

(i) the 'embedding' requires the extension of the coproduct, counit and antipode to formal series of generators belonging to the algebra G_q;

(ii) the deformed algebra G_q constructed from the following deformation scheme is *not* the same as the G_q defined in the Chevalley basis.

Let us point out what is the underlying idea for the proposed deformation scheme. Consider a semisimple Lie algebra G (of rank r) and a not regular maximal subalgebra (of rank l) $L \subset G$. For a classification and explicit construction of embeddings of semisimple Lie subalgebras see [6], where reference to the pioneering work of Dynkin on the subject can be found. The adjoint representation (ad_G) decomposes as

$$\mathrm{ad}_g \to \mathrm{ad}_L \oplus \mathrm{R}_L \tag{1.2}$$

where R_L is a representation, in general reducible, of L.

In the case of semi-simple Lie algebras, the algebra G can be constructed adding to the subalgebra L a suitable sets of elements belonging to R_L, e.g. in the case of $SU(3)$ one can add to the subalgebra $SO(3)$ a second-rank tensor operator highest-weight component. Then it is natural to wonder if a q-analogue of this procedure can be defined, i.e. to start by L_q and then to add some more suitable generators.

Let $\{E_i^{\pm}, H_i\}$ $(i=1, \ldots, l)$ be the generators of L in the Chevalley basis and $\{X_k^{\pm}, K_k\}$ $(k=1, \ldots, r-l)$ some elements of R_L with suitable properties. We call this basis L-basis as it depends on the choice of the subalgebra L. We remark that in all, at our knowledge, explicit realizations of the deformed algebras G_q the commuting elements are the same as the elements of the Cartan subalgebra of G. Then we define a deformation scheme in which the Cartan subalgebra of G, which is partly in the Cartan subalgebra of L, i.e. $\{H_i\}$, and partly in R_L, namely $\{K_k\}$, is left invariant and the set of $\{E_i^{\pm}, X_k^{\pm}\}$ is deformed in a suitable way. This deformation scheme will define a deformed algebra G_q which clearly *contains* the deformed algebra L_q, i.e. we have to build the chain

$$G_q \supset L_q. \tag{1.3}$$

Clearly some ambiguity is present in the definition of the deformation. In order to reduce the ambiguity we impose a *minimal deformation scheme* requiring:

(i) the Cartan subalgebra is *left unmodified* in the deformation;

(ii) if the commutator of two generators $g^+, g^- \in G$ gives an element k belonging to the Cartan subalgebra, than the commutator of the corresponding deformed generators gives $[k]_q$.

It has been shown in [4] that in the case of $SU_q(2)$ the q-tensor operators do not satisfy condition (i).

The deformation scheme we have just sketched requires that the generators of L_q be expressed in function of those of G_q. This is by no means evident *a priori*; we shall show that it can be really done in a number of examples by explicit constructions in terms of q-bosons and/or q-fermionic oscillators (see appendix) but it will be clear that our construction is based on the properties of the deformed oscillators and that not always a q-bosons and q-fermionic oscillators realization can be obtained even if in the case $q=1$ both are possible.

The aim of this paper is not to discuss in whole generality this formalism, but mainly to illustrate, in the case where the rank of L is one unit less the rank of G, the general features of the proposed deformation scheme which can be applied to other, even if not to any, specific chains one is interested in.

In section 2 we review the embedding of a singular subalgebra $L \subset G$ and introduce the L-basis for the algebra G. In section 3 we recall the definition of deformed Lie algebra G_q in the Chevalley basis and then extend the deformation procedure to the L-basis. In section 4 we present some examples of the deformed algebra in the L-basis, using explicit realizations of L_q and G_q in terms of q-bosons and/or q-fermionic oscillators. In section 5 a few conclusions, remarks and open questions are presented. Finally in the appendix we recall the definition of q-bosons and q-fermionic oscillators we use in the paper and we present a few useful q-identities.

2. The L-basis for Lie algebras

Let R be an irreducible representation (IR) of a subalgebra L of rank l of a Lie algebra G of rank r, see equation (1.2), and let V the carrier space of R.

In the following we need the following propositions.

Proposition 1. There exists a nonempty set $I \subset \{1, 2, \ldots, l\}$ such that for any $j \in I$ in V there is a vector (eventually degenerate) $X_j^{\pm} \in V$ which is eigenvector of the generators H_i spanning the Cartan subalgebra $H \subset L$ with the same eigenvalues as $E_j^{\pm}$, i.e. $\pm a_{ij}$ $(i = 1, \ldots, l)$.

Proof. The proposition follows from the remark that in R defined by (1.2) there are $r-l$ eigenvectors K_0 of H_i with zero eigenvalues and from the property that, if $\lambda=\{m_k\}$ is a weight those jth components is positive, then $\lambda'=\{m'_k=m_k-a_{kj}\}$ is also a weight.

In this paper we will consider only the simplest case where the rank of L is equal to $r-1$ where r is the rank of G. In this case there is only one element K_k, which we denote K_0, and only one not degenerate element X_k which we denote $X_j, j\subset\{1, 2, \ldots, l\}$.

With a suitable choice of normalization we can write

$$[K_0, E_j^{\pm}]=\pm X_j^{\pm} \tag{2.1}$$

$$[K_0, X_j^{+}]=E_j^{+}. \tag{2.2}$$

Proposition 2. If

$$[K_0, [K_0, E_j^{+}]]=[K_0, X_j^{+}]=E_j^{+} \tag{2.3}$$

then commutator of X_j^+ with X_j^- is equal to H_j.

Proof. From the Jacobi identities for the algebra G we have

$$[K_0, [E_j^-, X_j^+]]+[X_j^+, [K_0, E_j^-]]+[E_j^-, [X_j^+, K_0]]=0. \tag{2.4}$$

As the first commutator is vanishing as

$$[E_j^{\pm}, X_j^{\mp}]=\pm cK_0 \qquad (c\in C) \tag{2.5}$$

it follows

$$[K_0, [E_j^-, X_j^+]]=0=[X_j^+, X_j^-]+[E_j^-, E_j^+] \tag{2.6}$$

which implies

$$[X_j^+, X_j^-]=[E_j^+, E_j^-]=H_j. \tag{2.7}$$

Let us stress once more that (2.1) and (2.2) are not relations sufficient to define a Lie algebra structure on the set H_i, $E_i^{\pm}$, K_0, $X_i^{\pm}$. The fact the above set closes in a Lie algebra is a consequence of the fact it already belongs to a vectorial space G endowed with a Lie algebra structure, see equation (2.1). We start with a Lie algebra G defined in the Chevalley basis. Then a new basis is introduced which is formed by the generators of the subalgebra L (in the corresponding Chevalley basis) and by a tensor operator family of L (the set $K_0, X_i^{\pm}$). The new basis is a linear combination of the original generators of G. All this is trivial in absence of deformation. Then the generators of L and $X_i^{\pm}$ are deformed. *A priori* it is not clear that this deformation procedure can be really performed in a consistent way. The aim of this paper is to show that, at least in the simple cases considered, it can be performed and to analyse the results. Equations (2.1) and (2.2) are the 'minimum' set of relations we have to add to the definition of L in order to build up the algebra G. Finally let us also write the following proposition, even if we shall not use it in this paper, which may be useful in more general cases.

Proposition 3. If

$$[K_0, E_i^{+}]=0 \qquad (i\neq j)$$

then we have

$$(\mathrm{ad}\, E_i^+)^{1-a_{ij}} X_j^+ = 0 \qquad (i \neq j). \tag{2.8}$$

Proof. The Proposition follows from the following identity and recurrence formula

$$(\mathrm{ad}\, A)^n [B, C] = \sum_{0 \leqslant i \leqslant n} \begin{bmatrix} n \\ i \end{bmatrix} [(\mathrm{ad}\, A)^i B, (\mathrm{ad}\, A)^{n-i} C] \tag{2.9}$$

$$[(\mathrm{ad}\, A)^n B, C] = [A, [(\mathrm{ad}\, A)^{n-1} B, C] + [[A, C], (\mathrm{ad}\, A)^{n-1} B] \tag{2.10}$$

and from the Serre relations for the algebra L.

In the case $l = r-1$ we consider the following embeddings of singular maximal subalgebras L in the semi-simple Lie algebra G, of which we write also the corresponding decomposition of ad_G:

$A_1 \subset D_2$	$6 \to 3+3$
$A_1 \subset A_2$	$8 \to 3+5$
$A_1 \subset B_2$	$10 \to 3+7$
$C_2 \subset A_3$	$15 \to 10+5$
$G_2 \subset B_3$	$21 \to 14+7$
$B_{n-1} \subset D_n$	$n(2n-1) \to (n-1)(2n-1) + (2n-1)$.

We will discuss in section 4 all the corresponding deformations.

3. The deformation procedure

Herafter we shall use the same notations for the elements of the q-deformed algebras as for the non-deformed ones, hoping that no confusion arises. Let us recall the definition of G_q associated with a simple Lie algebra G of rank r defined by the Cartan matrix (a_{ij}) in the Chevalley basis. G_q is generated by $3r$ elements e_i^+, $f_i = e_i^-$ and h_i which satisfy $(i, j = 1 = 1, \ldots, r)$

$$\begin{aligned} &[e_i^+, e_j^-] = \delta_{ij} [\, j_i]_{q_i} \qquad &[h_i, h_j] = 0 \\ &[h_i, e_j^+] = a_{ij} e_j^+ \qquad &[h_i, e_j^-] = -a_{ij} e_j^- \end{aligned} \tag{3.1}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{3.2}$$

and $q_i = q^{d_i}$, d_i being non-zero integers with greatest common divisor equal to one such that $d_i a_{ij} = d_j a_{ji}$. Furthermore, the generators have to satisfy the Serre relations:

$$\sum_{0 \leqslant n \leqslant 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} (e_i^+)^{1-a_{ij}-n} e_j^+ (e_i^+)^n = 0 \tag{3.3}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q![n]_q} \qquad (3.4)$$

$$[n]_q! = [1]_q[2]_q \ldots [n]_q.$$

Analogous equations hold replacing e_i^+ by e_i^-. In the following we assume $h_i = (h_i)^+$ and the deformation parameter q to be real. The algebra G_q is endowed with a Hopf algebra structure. The action of the coproduct Δ, antipode S and co-unit ε on the generators is as follows:

$$\begin{aligned} &\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \qquad \Delta(e_i^\pm) = e_i^\pm \otimes q_i^{h_i/2} + q_i^{-h_i/2} \otimes e_i^\pm \\ &S(h_i) = -h_i \qquad S(e_i^\pm) = -q_i^{\mp 1} e_i^\pm \\ &\varepsilon(h_i) = \varepsilon(e_i^\pm) = 0 \qquad \varepsilon(1) = 1. \end{aligned} \qquad (3.5)$$

As the coproduct in a Hopf algebra satisfies ($g_i, g_j \in G_q$)

$$\Delta(g_i g_j) = \Delta(g_i)\Delta(g_j) \qquad (3.6)$$

it is essential to define which elements $\{g_i\}$ are the 'basis' of G_q.

Let us consider the algebra L_q defined in the Chevalley basis. Then the set $\{E_i^\pm, H_i\}$ ($i = 1, 2, \ldots, l$) (in the following we will denote the generators of L_q by capital letters) satisfy equations (3.1), (3.3) and (3.5). We introduce the set $(K_0, X_j^\pm)$ where

$$[K_0, E_j^\pm] = \pm X_j^\pm \qquad (3.7)$$

$$[H_j, X_j^\pm] = \pm a_{jj} X_j^\pm \qquad (3.8)$$

and we require

$$[X_j^+, X_j^-] = [H_j]_{qj} \qquad [H_i, K_0] = 0. \qquad (3.9)$$

The elements $(K_0, X_j^\pm)$ do not belong to L_q so *a priori* no coproduct or antipode or co-unit is defined on them. We extend the Hopf structure from L_q to $(K_0, X_j^\pm)$ as it follows:

$$\begin{aligned} &\Delta(K_0) = K_0 \otimes 1 + 1 \otimes K_0 \qquad \Delta(X_j^\pm) = X_j^\pm \otimes q_j^{H_j/2} + q_j^{-H_j/2} \otimes X_j^\pm \\ &S(K_0) = -K_0 \qquad S(X_j^\pm) = -q_j^{\mp 1} X_j^\pm \\ &\varepsilon(K_0) = \varepsilon(X_j^\pm) = 0. \end{aligned} \qquad (3.10)$$

Really we have to impose the Hopf structure only on the element K_0, the Hopf structure on $X_j^\pm$ can be derived from equations (3.5)–(3.8) the consistency of the coproduct being ensured by the equations (2.1) and (2.5). Let us emphasize once more that $\{H_i, K_0\}$, ($i = 1, \ldots, l-1$) are linear combinations of the elements of the basis of the Cartan subalgebra of G which are preserved unmodified in the deformation procedure.

4. Examples

In this section we will explicitly illustrate the proposed deformation procedure using an explicit realization of the sets $E_i^\pm, X_k^\pm$ in terms of q-deformed bosonic and/or fermionic oscillators, see appendix for notation.

It is well known that an explicit construction of $U_q(n)$ can be obtained by introducing q-analogue of the harmonic oscillator boson operator [7] satisfying a q-deformed Weyl algebra. Introducing also the q-analogue of the fermionic operators satisfying a q-deformed Clifford algebra [8, 1] this construction has been generalized to the q-universal enveloping algebra of all classical Lie algebra [1], to the exceptional Lie algebra [9, 10]. (Really in [10] the construction of $U_q(G_2)$ has been obtained in terms of q-quasiparafermions (called 'q-skedofermions'), a realization of $U_q(G_2)$ in terms of q-fermions is given in [11].)

4.1. $SO_q(4) \supset SO_q(3)$

Let us consider the six generators J^α_μ, ($\alpha = 1, 2; \mu = \pm, 0$) defining two commuting $SU(2)$. If we deform these generators and impose on them or on their sum and difference the Hopf structure we get $SO_q(4)$, where $SO_q(3)$ is no more embedded. In order to preserve the 'embedding' of $SO(3)$ we proceed according to the scheme of section 3. We deform the generators J^α_μ, i.e. we require ($\alpha, \beta = 1, 2$):

$$[J^\alpha_+, J^\beta_-] = \delta_{\alpha,\beta}[2J^\alpha_0]_q \qquad [J^\alpha_0, J^\beta_\pm] = \pm\delta_{\alpha,\beta}J^\alpha_\pm \tag{4.1}$$

then we define

$$L_\pm = J^1_\pm q^{J^2_0} + J^2_\pm q^{-J^1_0} \tag{4.2}$$

$$L_0 = J^1_0 + J^2_0 \tag{4.3}$$

which satisfy

$$[L_+, L_-] = [2L_0]_q \qquad [L_0, L_\pm] = \pm L_\pm . \tag{4.4}$$

We impose the Hopf structure on the elements L_μ so getting $SO_q(3)$. Then we add the element

$$K_0 = J^1_0 - J^2_0 \tag{4.5}$$

and define

$$[K_0, L_\pm] = \pm X_\pm \qquad [X_+; X_-] = 2L_0 \tag{4.6}$$

where

$$X_\pm = J^1_\pm q^{J^2_0} - J^2_\pm q^{-J^1_0}. \tag{4.7}$$

Clearly we have

$$[X_+, X_-] = [2L_0]_q \qquad [L_0, X_0] = 0 \qquad [L_0, X_\pm] = \pm X_\pm . \tag{4.8}$$

We impose a coproduct, antipode and co-unit on K_0, see equation (3.9), and then we consider the set of elements $\{L_\mu, X_\pm, K_0\}$ as spanning $SO_q(4)$ in the $SO(3)$ basis. Of course in the limit $q = 1$ we recover $SO(4)$. Let us remark that the coproduct defined on $L_\pm$ is not equivalent to the coproduct defined on the elements $J^\alpha_\pm$ in the Chevalley basis, see [12]. The generators of $SO_q(4)$ in the Chevalley basis, denoted by $\{J_p, N_p; p = \pm, 3\}$ in [12], are related to our generators by the following relations:

$$J_3 = L_0 \qquad N_3 = K_0 \tag{4.9}$$

$$J^1_\pm = q^{-J^2_0}(L_\pm - X_\pm)/2 \tag{4.10}$$

$$J^2_\pm = q^{-J^1_0}(L_\pm - X_\pm)/2. \tag{4.11}$$

Let us also remark that our construction is *not invariant* for $q \to q^{-1}$ due to the q factors appearing in the definition of $L_{\pm}$. However an invariant construction can be obtained, assuming that the generators $\{J^a_\mu\}$ are invariant, defining

$$L_{\pm} = J^1_{\pm}(q^{J^2_0} + q^{-J^2_0})/2 + J^2_{\pm}(q^{J^1_0} + q^{-J^1_0})/2. \tag{4.12}$$

To simplify the formulae hereafter we shall use the not invariant formulation.

Let us give now an explicit realization in terms of q-bosons and q-fermionic oscillators.

I. q-Bosons realization

Let us introduce four q-bosons $\{b_i^+, b_i\}$ (i = 1, 2, 3, 4). We then can write

$$L_+ = b_1^+ b_2 q^{(N_3 - N_4)/2} + b_3^+ b_4 q^{-(N_1 - N_2)/2} \qquad L_- = (L_+)^+ \tag{4.13}$$

$$L_0 = \tfrac{1}{2}(N_1 - N_2 + N_3 - N_4) \qquad K_0 = \tfrac{1}{2}(N_1 - N_2 - N_3 + N_4) \tag{4.14}$$

$$X_+ = b_1^+ b_2 q^{(N_3 - N_4)/2} - b_3^+ b_4 q^{-(N_1 - N_2)/2}. \tag{4.15}$$

In the Fock space of the q-bosons we can build representations of integer angular momentum provided we consider states which are obtained from the vacuum state by an equal number of creation operators of the set $\{b_j^+, j = 1, 2\}$ and of the set $\{b_k^+, k = 3, 4\}$.

II. q-Fermionic oscillators realization

Introducing a set of two q-fermionic oscillators we can write

$$L_+ = a_1^+ a_2 q^{(M_1 + M_2)/2} + a_1^+ a_2^+ q^{-(M_1 - M_2)/2} \qquad L_- = (L_+)^+ \tag{4.16}$$

$$L_0 = M_1 \qquad K_0 = -M_2. \tag{4.17}$$

In this construction we cannot get the integer angular momentum states in the Fock space of the q-fermionic oscillators due to the vanishing of the square of a fermionic oscillator. Clearly in the limit of $q = 1$ the set $(X_{\pm}, K_0)$ transform under the $SO(3)$ spanned by $\{L_\mu\}$ as a vector representation.

4.2. $SU_q(3) \supset SO_q(3)$

I. q-Bosons realization

$SO_q(3)$ [3] is defined by

$$[L_+, L_-] = [2L_0]_q \qquad [L_0, L_{\pm}] = \pm L_{\pm}. \tag{4.18}$$

This $SO_q(3)$ can be endowed with an Hopf structure in the standard way, see [4]. Introducing a set of three q-boson b_μ, $b_\mu^\dagger$, $(\mu = 1, 0, -1)$ a realization of $SO_q(3)$ can be written:

$$L_0 = N_1 - N_{-1} \tag{4.19}$$

$$L_+ = q^{N_{-1}} q^{-1/2N_0} \sqrt{q^{N_1} + q^{-N_1}}\, b_1^+ b_0 + b_0^+ b_{-1} q^{N_1} q^{-1/2N_0} \sqrt{q^{N_{-1}} + q^{-N_{-1}}} \tag{4.20}$$

$$L_- = b_0^+ b_1 q^{N_{-1}} q^{-1/2N_0} \sqrt{q^{N_1} + q^{-N_1}} + q^{N_1} q^{-1/2N_0} \sqrt{q^{N_{-1}} + q^{-N_{-1}}}\, b_{-1}^+ b_0. \tag{4.21}$$

Let us remark that L_+, L_- are *not invariant* for $q \to q^{-1}$, while L_0 is invariant. The dimension of the representations of $SO_q(3)$ in the Fock space of the q-boson oscillators is given by $2N+1$ ($N = n_1 + n_0 + n_{-1}$) therefore it is always *odd*. So this is a realization of $SO_q(3)$ and not os $SU_q(2)$. Moreover for $q=1$ we obtain the usual $SO(3)$ realization in terms of boson oscillators.

Let us add to the generators of $SO_q(3)$ the operator

$$K_0 = \tfrac{1}{3}(2N_0 - N_1 - N_{-1}) \tag{4.22}$$

which satisfy

$$[K_0, L_0] = 0 \qquad [K_0, L_\pm] = \pm X_\pm \tag{4.23}$$

where $X_+ (X_- = (X_+)^+)$ is given by

$$X_+ = -q^{N_{-1}} q^{-1/2N_0} \sqrt{q^{N_1} + q^{-N_1}}\, b_1^+ b_0 + b_0^+ b_{-1} q^{N_1} 1^{-1/2N_0} \sqrt{q^{N_{-1}} + q^{-N_{-1}}}. \tag{4.24}$$

Then we have

$$[K_0, X_\pm] = \pm L_\pm \qquad [X_+, X_-] = [2L_0]_q. \tag{4.25}$$

One can gain a better insight on the meaning of the elements K_0 and $X_\pm$ in the limit $q=1$, where they become (up to a multiplicative factor) respectively the $m=0$ and the $m=\pm 1$ component of a rank-2 tensor operator for $SO(3)$.

II. q-Fermionic oscillators realization

We can also write a realization of $SO_q(3)$ in terms of a set of three q-fermionic oscillators just replacing in the above formulae b_μ by a_μ and N_μ by M_μ. However in the Fock space of q-fermionic oscillators we can realize only the three-dimensional vector representation.

The relations between the generators of $U_q(3)$ in the Chevalley basis [3], denoted by h_i, $e_i^\pm$ ($e_i^- = (e_i^+)^+$) ($i = 1, 2$), and our generators are:

$$h_1 = \tfrac{1}{2}(L_0 - 3K_0) \qquad h_2 = \tfrac{1}{2}(3K_0 + L_0) \tag{4.26}$$

$$e_1^+ = \tfrac{1}{2} q^{-N_{-1}} q^{1/2N_0} (q^{N_1} + q^{-N_1})^{-1/2} (L_+ - X_+) \tag{4.27}$$

$$e_2^+ = \tfrac{1}{2}(L_+ + X_+) q^{-N_1} q^{1/2N_0} (q^{N_{-1}} + q^{-N_{-1}})^{-1/2} \tag{4.28}$$

$$h_3 = N_0 + N_1 + N_{-1} = L_0 + 2K_0 + 3N_{-1}. \tag{4.29}$$

From (4.27)–(4.28) it follows that the definition, for instance, of the coproduct on the elements $e_i^\pm$ from the coproduct on ($L_\pm, X_\pm, h_1, h_2, h_3$) requires the extension of (2.6) to formal series expansion of $(q^{N_{\pm 1}} + q^{-N_{\pm 1}})^{-1/2}$. This feature will be present in all the following formulae.

4.3. $Sp_q(4) \supset SO_q(3)$

I. q-Bosons realization

Introducing a set of two q-boson (b_1, b_1^+, b_2, b_2^+) a realization of $SO_q(3)$ can be written:

$$L_0 = (-3N_1 - N_2 - 2)/2 \tag{4.30}$$

$$L_+ = q b_1 b_2^+ \sqrt{q^{N_1} + q^{-N_1} + 1}\, q^{N_2} + q^{-1} (b_2)^2 q^{1/2(-3N_1 + N_2)} \tag{4.31}$$

$$L_- = q^{N_2} q \sqrt{q^{N_1} + q^{-N_1} + 1}\, b_2 b_1^+ - q^{-1} q^{1/2(-3N_1 + N_2)} (b_2^+)^2 \tag{4.32}$$

with

$$[L_+, L_-] = [2L_0]_q. \tag{4.33}$$

Also in this case $L_\pm$, are *not invariant* for $q \to q^{-1}$, while L_0 is invariant.

Let us add to the generators of $SO_q(3)$ the operator

$$K_0 = \tfrac{1}{2}(N_2 - N_1) \tag{4.34}$$

which satisfy

$$[K_0, L_0] = 0 \qquad [K_0, L_\pm] = \pm X_\pm \tag{4.35}$$

where $X_\pm$ are given by

$$X_+ = q b_1 b_2^+ \sqrt{q^{N_1} + q^{-N_1} + 1}\, q^{N_2} - q^{-1}(b_2)^2 q^{1/2(-3N_1+N_2)} \tag{4.36}$$

$$X_- = q^{N_2} q \sqrt{q^{N_1} + q^{-N_1} + 1}\, b_2 b_1^+ + q^{-1} q^{1/2(-3N_1+N_2)} (b_2^+)^2. \tag{4.37}$$

Then we have

$$[K_0, X_\pm] = \pm L_\pm \qquad [X_+, X_-] = [2L_0]_q. \tag{4.38}$$

The relation between the generators of $Sp_q(4)$ in the Chevalley basis [1] and our generators are (1 'short root', 2 'long root'):

$$h_1 = 2K_0 = N_2 - N_1 \qquad h_2 = \tfrac{1}{2}(L_0 - 3K_0) = -N_2 - \tfrac{1}{2} \tag{4.39}$$

$$e_1^+ = \tfrac{1}{2} q^{-1}(L_+ + X_+)(q^{N_1} + q^{-N_1} + 1)^{-1/2} q^{-N_2} \tag{4.40}$$

$$e_2^+ = \tfrac{1}{2}(q + q^{-1})^{-1} q (L_+ - X_+) q^{1/2(3N_1 - N_2)} \tag{4.41}$$

$$e_i^- = (e_i^+)^+ \qquad (i = 1, 2). \tag{4.42}$$

II. q-Fermionic oscillators realization

From the isomorphism between $Sp(4)$ and $SO(5)$ we can construct a q-fermion realization of $SO_q(5) \supset SO_q(3)$.

Introducing a set of 2 q-fermionic oscillators (a_1, a_1^+, a_2, a_2^+) a realization of $SO_q(3)$ can be written $(q = e^\lambda)$:

$$\begin{aligned} L_+ = a_1^+ a_2 \{2\cosh \lambda(3M_1 + 3M_2 - 3) + (q^2 + q^{-2}) \cosh \lambda(M_1 + M_2 - 1)\}^{1/2} \\ + A(2)_2^+ \{(q + q^{-1}) \cosh \lambda(2M_1 + M_2 - 1) + \tfrac{1}{2}(q^2 + q^{-2})\}^{1/2} \end{aligned} \tag{4.43}$$

$$L_- = (L_+)^+ \qquad L_0 = 2M_1 + M_2 - \tfrac{3}{2}$$

where

$$A(n)_i A(n)_j^+ + q^{n\delta_{ij}} A(n)_j^+ A(n)_i = \delta_{ij} q^{M_i} \tag{4.44}$$

$$[M_i, A(n)_j^+] = \delta_{ij} A(n)_j^+ \qquad [M_i, A(n)_j] = -\delta_{ij} A(n)_j \qquad [M_i, M_j] = 0. \tag{4.45}$$

See, however, the appendix for comments on the q-fermionic oscillators. The construction is invariant for $q \to q^{-1}$.

We add to the generators of $SO_q(3)$ the operator

$$K_0 = -M_2 \tag{4.46}$$

and define

$$[K_0, E_\pm] = \pm X_\pm \tag{4.47}$$

and then we proceed as before. We can also express in a straightforward way the generators of $SO_q(5)$ in the Chevalley basis, [1] in terms of $(L_\pm, X_\pm, L_0, K_0)$.

4.4. $SU_q(4) \supset Sp_q(4)$

Introducing a set of four q-fermionic oscillators we can write:

$$E_1^+ = a_1^+ a_2 q^{(M_3 - M_4)/2} + q^{-(M_1 - M_2)/2} a_3^+ a_4 \tag{4.48}$$

$$E_2^+ = (q^{M_2} + q^{-M_2})^{1/2} a_2^+ a_3 (q^{M_3} + q^{-M_3})^{1/2} (q + q^{-1})^{-1} \tag{4.49}$$

$$E_i^- = (E_i^+)^+ \qquad (i = 1, 2) \tag{4.50}$$

$$H_1 = M_1 - M_2 + M_3 - M_4 \qquad H_2 = M_2 - M_3 \tag{4.51}$$

which satisfy

$$[E_1^+, E_1^-] = [H_1]_q \qquad [E_2^+, E_2^-] = [H_2]_{q^2}. \tag{4.52}$$

We add the following element

$$K_0 = M_2 + M_3. \tag{4.53}$$

Then we have

$$[K_0, H_i] = 0 \qquad [K_0, E_2^\pm] = 0 \qquad [K_0, E_1^\pm] = \pm X_1^\pm \tag{4.54}$$

where

$$X_1^+ = -a_1^+ a_2 q^{(M_3 - M_4)/2} + q^{-(M_1 - M_2)/2} a_3^+ a_4 \tag{4.55}$$

and

$$[K_0, X_1^\pm] = \pm E_1^\pm \qquad [X_1^+, X_1^-] = [H_1]_q. \tag{4.56}$$

The above construction is *not invariant* for $q \to q-1$, but it is possible to write an invariant one along the same lines discussed in subsection 4.1. The Serre relations are trivially satisfied due to the vanishing of the square of a q-fermion.

Let us remark that an analogue q-boson realization *cannot be obtained* as we get

$$[E_1^+, E_2^-] \neq 0 \tag{4.57}$$

and moreover the Serre relations are not satisfied.

The relations between the generators of $U_q(4)$ in the Chevalley basis [1] and our generators are $(f_i = e_i^+ ; i = 1, 2, 3)$:

$$h_1 = H_1 + \tfrac{1}{2}(H_2 - K_0) + M_4 = M_1 - M_2 \qquad h_2 = H_2 \tag{4.58}$$

$$h_3 = \tfrac{1}{2}(K_0 - H_2) - M_4 = M_3 - M_4 \qquad h_4 = H_1 + H_2 + K_0 + 2M_4 \tag{4.59}$$

$$e_1 = \tfrac{1}{2}(E_1^+ - X_1^+) q^{(N_3 - M_4)/2} \tag{4.60}$$

$$e_2 = (q^{M_2} + q^{-M_2})^{-1/2} E_2^+ (q^{M_3} + 1^{-M_3})^{-1/2} (q + q^{-1}) \tag{4.61}$$

$$e_3 = \tfrac{1}{2} q^{(M_1 - M_2)/2} (E_1^+ + X_1^+). \tag{4.62}$$

4.5. $SO_q(7) \supset U_q(G_2)$

We use the following realization [11] of the algebra $U_q(G_2)$, *not invariant* for $q \to 1^{-1}$, in terms of q-fermionic oscillators:

$$E_1^+ = A(3)_1^+ A(3)_2$$

$$E_2^+ = 1^{-1/2(M_3 - M_1)} \sqrt{\frac{q^{(M_2-1)} + q^{-(M_2-1)}}{2}}\, a_2^+ + q^{(M_2-1/2)} a_3^+ a_1 \tag{4.63}$$

$$E_i^- = (E_i^+)^+ \qquad (i = 1, 2) \tag{4.64}$$

we get

$$[E_1^+, E_1^-] = [H_1]_{q^3} = [M_1 - M_2]_{q^3} \tag{4.65}$$

$$[E_2^+, E_2^-] = [H_2]_q = [2M_2 + M_3 - M_1 - 1]_q \tag{4.66}$$

where we have used the q-identities given in the appendix.

An invariant construction can be obtained modifying the definition of E_2^+ as in subsection 4.1.

We add the following element

$$K_0 = M_1 + M_2 \tag{4.67}$$

then we have

$$[K_0, H_i] = [K_0, E_1^\pm] = 0 \qquad [K_0, E_2^\pm] = \pm X_2^\pm \tag{4.68}$$

where

$$X_2^+ = q^{-1/2(M_3 - M_1)} \sqrt{\frac{q^{(M_2-1)} + q^{-(M_2-1)}}{2}}\, a_2^+ - q^{(M_2-1/2)} a_3^+ a_1 \tag{4.69}$$

and

$$[K_0, X_2^\pm] = \pm E_2^\pm \qquad [X_2^+, X_2^-] = [H_2]_q . \tag{4.70}$$

4.6. $SO_q(2n) \supset SO_q(2n-1)$

We write the following realization for $SO_q(2n-1)$:

$$(k = 1, \ldots, n-2)$$

$$E_k^+ = A(2)_k^+ A(2)_{k+1} \tag{4.71}$$

$$E_{n-1}^+ = a_{n-1}^+ a_n q^{(M_{n-1}+M_n)/2} + q^{-(M_{n-1}-M_n)/2} a_{n-1}^+ a_n^+ \tag{4.72}$$

$$E_i^- = (E_i^+)^+ \qquad (i = 1, \ldots, n-1) \tag{4.73}$$

$$H_k = M_k - M_{k+1} \qquad H_{n-1} = 2M_{n-1} . \tag{4.74}$$

The above construction is *not invariant* for $q \to q^{-1}$, but it can be made invariant, as in subsection 4.1. It is easy to verify that the above realization satisfies (3.1), with the Cartan matrix of $SO(2n-1)$, the Serre relations (14) being verified in a trivial way due to the vanishing of the square of a fermionic oscillator. In particular we have:

$$[E_k^+, E_k^-] = [H_k]_{q^2} \qquad [E_{n-1}^+, E_{n-1}^-] = [H_{n-1}]_q . \tag{4.75}$$

Then we add the element

$$K_0 = M_n \tag{4.76}$$

which satisfies

$$[K_0, H_i] = 0 \qquad [K_0, E_k^{\pm}] = 0 \qquad [K_0, E_{n-1}^{\pm}] = \pm X_{n-1}^{\pm} \tag{4.77}$$

where

$$X_{n-1}^{+} = -a_{n-1}^{+} a_n q^{(M_{n-1}+M_n)/2} + q^{-(M_{n-1}-M_n)/2} a_{n-1}^{+} a_n^{+}. \tag{4.78}$$

Then we have:

$$[K_0, X_{n-1}^{\pm}] = \pm E_{n-1}^{\pm} \qquad [X_{n-1}^{+}, X_{n-1}^{-}] = [H_{n-1}]_q. \tag{4.79}$$

An insight on the above realization is obtained in the limit $q=1$. In this limit the realization of $SO(2n-1)$ is just the realization obtained by 'folding' the fermion realization of $SO(2n)$ and K_0 is the vector of the fundamental vectorial representation with vanishing eigenvalues with respect to the Cartan subalgebra of $SO(2n-1)$. Let us emphasize that in the above realization the role of the q-fermionic oscillators is crucial.

5. Conclusions

We have presented a deformation scheme in a basis different from the Chevalley basis which allows us to build up embedding chains of q-algebras which may be useful in application in physics. At the end of each subsections of section 4 we have explicitly written the relations between the generators in the Chevalley basis of the deformed algebras and the generators of the 'deformed' algebra in the L-basis. The definition of the coproduct allows one to perform tensor product of spaces which is an essential requirement to almost all applications one can imagine. Let us emphasize once more that the word 'embedding' should be taken in the loose sense we have specified in section 1. In fact the deformed Lie algebra G obtained is equivalent as enveloping algebra to the standard deformed G_q, i.e. the one defined in the Chevalley basis by (3.1)–(3.5), but it is endowed by a Hopf structure different from the one of G_q. We believe however that, according to the problem one is dealing with, it can be useful to use the L-basis or the Chevalley basis, in which, e.g. the R universal matrix has been built up. Let us emphasize that in this deformation scheme an extension of the definition of the coproduct from the enveloping algebra to a formal infinite series it is necessary due to the appearance in the equations of a square root expression of the sum of the exponentials of the N_i or M_i. A peculiar feature of this deformation scheme is the fact that the deformed algebra L_q is not always invariant for $q \to q^{-1}$; only the commuting subalgebra is always invariant. It seems that when the subalgebra L of G (for $q=1$) is obtained by taking linear combination of generators corresponding to non-connected roots, then one can build up invariant realization of L_q. Another peculiar feature is the fact that in the L-basis we need the q-Serre relations only on the subalgebra L (in the cases considered in the paper two relations less than in the Chevalley basis). Indeed once the deformed set of generators $\{E_i^{\pm}, K_0; i=1, \ldots, l\}$, satisfying (3.1) and (3.3), are introduced the properties of the elements $\{X_j^{\mp}\}$ are uniquely defined by (3.7). However, let us emphasize that the choice of the element K_0 requires the knowledge of the whole algebra G, while for the deformation in the Chevalley basis only the knowledge of the generators corresponding to the simple roots of G is required. In this paper we have restricted our analysis to the simplest case in which L is a simple maximal non-regular subalgebra of rank $r-1$ of G and R is an irreducible representation. Many open problems are still present. In particular the choice of the set of elements K_k is some way arbitrary and it is not evident that the method can be applied to the deformation of any embedding of any semi-simple Lie algebra L or to any embedding chain as

$$G_q \supset L_q \supset J_q$$

It is also worth emphasizing that

(i) the construction of the subalgebra L_q is not unique, e.g. for another form of $SO_q(3)$ see [4, 13];

(ii) in our explicit construction, which exploits the property of factorization of the product of two generators, the role of q-bosons and q-fermionic oscillators is essential.

In the light of possible applications one should also build up invariant operators (the q-analogue of the Casimir operators) in the above defined basis. It is also an interesting problem to study how the representations of G_q decompose with respect to L_q; in this context the choice of the definition of the coproduct is relevant.

We believe that it is possible to generalize this method to the real forms of the deformed simple Lie algebra and, moreover, to build up deformation scheme for non-semi-simple Lie algebra.

Appendix

Let us recall, to fix the notation, the definition of q-bosons which we denote by b_i^+, b_i and q-fermionic oscillators which we denote by a_i^+, a_i:

$$b_i b_j^+ - q^{\delta_{ij}} b_j^+ b_i = \delta_{ij} q^{-N_i} \tag{A.1}$$

$$[N_i, b_j^+] = \delta_{ij} b_j^+ \qquad [N_i, b_j] = -\delta_{ij} b_j \qquad [N_i, N_j] = 0 \tag{A.2}$$

$$a_i a_j^+ + q^{\delta_{ij}} a_j^+ a_i = \delta_{ij} q^{M_i} \tag{A.3}$$

$$[M_i, a_j^+] = \delta_{ij} a_j^+ \qquad [M_i, a_j] = -\delta_{ij} a_j \qquad [M_i, M_j] = 0. \tag{A.4}$$

It may be useful to recall the following identities:

$$b_i^+ b_i = \frac{q^{N_i} - q^{-N_i}}{q - q^{-1}} \qquad b_i b_i^+ = \frac{q^{N_i+1} - q^{-N_i-1}}{q - q^{-1}} \tag{A.5}$$

$$a_i^+ a_i = \frac{q^{M_i} - q^{-M_i}}{q - q^{-1}} \qquad a_i a_i^+ = \frac{q^{-M_i+1} - q^{M_i-1}}{q - q^{-1}}. \tag{A.6}$$

The q-bosons are assumed to commute with the q-fermionic oscillators.

We shall need the following useful identities:

$$(q^{nA} - q^{-nA}) = (q^A - q^{-A}) \times P(q, n, A) \tag{A.7}$$

where

$$P(q, n, A) = \sum_{0 \leqslant i \leqslant [n-2/2]} [(q^{(n-1-2i)A} + q^{-(n-1-2i)A}) + (1 - (-1)^n)/2] \tag{A.8}$$

$$P(q \to 1, n, A) \to n \tag{A.9}$$

$$[X]_q q^{-Y} + [Y]_q q^X = [X+Y]_q \tag{A.10}$$

$$[nX]_q [n(Y+1)]_q - [n(X+1)]_q [nY]_q = [n(X-Y)]_q P(q, n, 1) \tag{A.11}$$

where A, X, Y are operators ($[X, Y]=0$) or c-numbers, n is an integer positive number ($2\leqslant n$) and $[n-2/2]$ denotes the integer part of $(n-2)/2$. $P(q, n, A)$ is invariant for $q\to q^{-1}$.

From (A.8) we can define bosonic *dilation* operators

$$D_i^b(n)=\sqrt{P(q,n,N_i)/P(q,n,1)} \tag{A.12}$$

acting on the q-bosons such that if the q-bosons b_i^+ and b_j satisfy (A.1) then the q-bosons

$$B_j^+(n)=D_j^b(n)b_j^+ \qquad B_i(n)=b_iD_i^b(n) \tag{A.13}$$

satisfy

$$B_i(n)B_j^+(n)-q^{n\delta_{ij}}B_j^+(n)B_i(n)=\delta_{ij}q^{-nN_i} \tag{A.14}$$

and

$$[N_i, B_j^+(n)]=\delta_{ij}B_j^+(n) \qquad [N_i, B_j(n)]=-\delta_{ij}B_j(n) \qquad \forall n. \tag{A.15}$$

In [14] the explicit construction of q-bosons in terms of non-deformed standard bosonic oscillators has been given. From this construction we can build up dilation operators which hold for any value of q' ($q=\exp\lambda$, $q'=\exp\lambda'$):

$$D_i^b(q'/q)=[\sinh\lambda' N_i/N_i\sinh\lambda']^{1/2}\times[\sinh\lambda N_i/N_i\sinh\lambda]^{-1/2}. \tag{A.16}$$

It is also known that, due to the fact that the square of the fermionic number operator for non-deformed as well as for the deformed fermionic oscillators is equal to the number operator itself, equations (A.3), (A.4) are satisfied by standard fermionic oscillators. In [15] a definition of q-fermion operators *not equivalent* to the usual fermion operators is given by the equations

$$\psi_i\psi_j^+ + q^{\delta_{ij}}\psi_j^+\psi_i=\delta_{ij}q^{-\Omega_i} \tag{A.17}$$

$$[\Omega_i, \psi_j^+]=\delta_{ij}\psi_j^+ \qquad [\Omega_i, \psi_j]=-\delta_{ij}\psi_j. \tag{A.18}$$

The authors have shown that for $q\neq 1$ the square of the q-fermion $\psi(\psi^+)$ vanishes *only* on the state of the Fock space. The above equation *is not invariant* for $q\to q^{-1}$. Should this invariance be assumed in the limit $q=1$ one could find the number operator (Ω) equal to $-\psi\psi^+$. In order to build up realization of deformed Lie algebras in terms of oscillators one needs to know the value of bilinears of the type $\psi_i\psi_i^+$, see (A.5) and (A.6), which cannot be computed from (A.17) without any further assumption. In fact the authors in [15] have built in terms of q-fermions a realization of $SU_q(2)$ *only* on the states of the Fock space. Such a restriction cannot be a very serious one for physical applications, so an investigation of the possibility to realize general deformed Lie algebras in terms of q-fermions should be carried on. In the present paper we have limited to consider the q-deformed fermionic oscillators given by (A.3) and (A.4) in terms of which, as shown in [1], the deformed Lie of the series A_n, B_n, D_n can be realized as bilinears of fermionic oscillators. For a discussion of the different deformed fermionic algebras see [16].

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